

# CLASSIFICATION OF NON-SYMPLECTIC AUTOMORPHISMS OF ORDER 3 ON $K3$ SURFACES

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ABSTRACT. In this paper, we study non-symplectic automorphisms of order 3 on algebraic  $K3$  surface over  $\mathbb{C}$  which act trivially on the Néron-Severi lattice. In particular we shall characterize their fixed locus in terms of the invariants of 3-elementary lattices.

## 1. INTRODUCTION

Let  $X$  be an algebraic surface over  $\mathbb{C}$ . If its canonical line bundle  $K_X$  is trivial and  $\dim H^1(X, \mathcal{O}_X) = 0$  then  $X$  is called a  $K3$  surface. In this paper, we study automorphisms of algebraic  $K3$  surfaces.

In the following, for an algebraic  $K3$  surface  $X$ , we denote by  $S_X$ ,  $T_X$  and  $\omega_X$  the Néron-Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form on  $X$ , respectively.

An automorphism of  $X$  is *symplectic* if it acts trivially on  $\mathbb{C} \cdot \omega_X$ . In particular, this paper is devoted to study of *non-symplectic* automorphisms on algebraic  $K3$  surfaces.

First we recall some general results about non-symplectic automorphisms on algebraic  $K3$  surfaces. Nikulin [Ni3] has classified non-symplectic involutions by using classification of 2-elementary lattices. In the paper, he considered fixed locus of involutions. And he characterize it in terms of the invariants of 2-elementary lattices.

More generally, we suppose that  $g$  is a non-symplectic automorphism of order  $I$  on  $X$  such that  $g^*\omega_X = \zeta_I\omega_X$  where  $\zeta_I$  is a primitive  $I$ -th root of unity. Then  $g^*$  has no non-zero fixed vectors in  $T_X \otimes \mathbb{Q}$  and hence  $\phi(I)$  divides  $\text{rank } T_X$ , where  $\phi$  is the Euler function. In particular  $\phi(I) \leq \text{rank } T_X$  and hence  $I \leq 66$  ([Ni2], Theorem 3.1 and Corollary 3.2).

The extremal case  $\text{rank } T_X = \phi(I)$  is determined in the result below [Vo], [Kon1]. If  $T_X$  is unimodular then  $I \in \{66, 44, 42, 36, 28, 12\}$ . Moreover for  $I \in \{66, 44, 42, 36, 28, 12\}$ , there exist a unique pair  $(X_I, \langle g_I \rangle)$

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a  $K3$  surface and an order  $I$  automorphisms group. And if  $T_X$  is not unimodular and  $g_I$  acts trivially on  $S_X$  then  $I \in \{3^k, 5^l, 7, 11, 13, 17, 19\}$  where  $1 \leq k \leq 3$  and  $l = 1, 2$ .

Non-symplectic automorphisms with order 13, 17 and 19 are studied by Oguiso and Zhang [OZ2]. They proved that there exist a unique pair  $(X_I, \langle g_I \rangle)$  of a  $K3$  surface and an order  $I$  automorphisms group where  $I \in \{13, 17, 19\}$ . In particular,  $(X_I, \langle g_I \rangle)$  are isomorphic to Kondo's example in [Kon1].

Sometimes an automorphism with small order can also characterize uniquely a  $K3$  surface. Actually non-symplectic automorphisms with order 3 are studied by Oguiso and Zhang [OZ1]. They proved that there exist a unique pair  $(X, g)$  of a  $K3$  surface and an order 3 automorphism such that the fixed locus  $X^g = \{x \in X | g(x) = x\}$  consists of only at least 6 rational curves and some isolated points.

Following the Nikulin's result [Ni3], we characterize fixed locus in terms of the invariants of 3-elementary lattices.

Let  $\rho$  be the Picard number of  $X$ , let  $S_X^* = \text{Hom}(S_X, \mathbb{Z})$ , and let  $s$  be the minimal number of generators of  $S_X^*/S_X$ .

**Proposition 1.1** ([Vo], [Kon1]). Assume that there exists a non-symplectic automorphism  $\varphi$  of order  $p$  on  $X$  which act trivially on  $S_X$ . Then  $S_X$  is a  $p$ -elementary lattice, i.e.  $S_X^*/S_X$  is  $p$ -elementary abelian group.

Then in the following we assume that  $S_X$  is 3-elementary. The main purpose of this paper is to prove the following theorem:

**Theorem 1.2.** We assume that  $S_X$  is 3-elementary.

- (1) If  $22 - \rho - 2s < 0$ , then  $X$  has no non-symplectic automorphism of order 3 which acts trivially on  $S_X$ .
- (2) If  $22 - \rho - 2s \geq 0$ , then  $X$  has a non-symplectic automorphism  $\varphi$  of order 3 which act trivially on  $S_X$ . Moreover the fixed locus  $X^\varphi := \{x \in X | \varphi(x) = x\}$  has the form

$$X^\varphi = \begin{cases} \{P_1\} \amalg \{P_2\} \amalg \{P_3\} & \text{if } S_X = U(3) \oplus E_6^*(3) \\ \{P_1\} \amalg \cdots \amalg \{P_M\} \amalg C^{(g)} \amalg E_1 \amalg \cdots \amalg E_{N-1} & \text{otherwise} \end{cases}$$

and  $M = \rho/2 - 1$ ,  $g = (22 - \rho - 2s)/4$ ,  $N = (6 + \rho - 2s)/4$ , where we denote by  $P_i$  an isolated point,  $C^{(g)}$  a non-singular curve of genus  $g$  and by  $E_j$  a non-singular rational curve.

**Remark 1.3.** In case if  $22 - \rho - 2s \geq 0$  then there exist examples of  $K3$  surfaces with non-symplectic automorphism of order 3 which act trivially on  $S_X$ . (See section 5.)

Moreover we shall give Table 2 in the last of section 4 in which the fixed locus  $X^\varphi$  is given for each  $S_X$  satisfying  $22 - \rho - 2s \geq 0$ .

In section 2, we shall give a classification of an even hyperbolic 3-elementary lattices admitting a primitive imbedding in  $K3$  lattice. In section 3, we characterize existence of isometries of order 3 by invariants of 3-elementary lattice. The main part in this paper is Section 4. In this section, we shall prove the theorem. Here we use mainly the Lefschetz formula, the Hurwitz formula and the theory of elliptic surfaces due to Kodaira [Kod]. In section 5, we give examples of algebraic  $K3$  surfaces with non-symplectic automorphism of order 3 which act trivially on  $S_X$ .

**Remark 1.4.** Recently, M. Artebani, A. Sarti [AS] independently have given a classification of non-symplectic automorphisms of order 3 on  $K3$  surfaces by using Smith exact sequences. Our proof is based on geometric argument without using Smith exact sequences.

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## 2. CLASSIFICATION OF 3-ELEMENTARY LATTICES ADMITTING A PRIMITIVE IMBEDDING IN $K3$ LATTICE

A lattice of rank  $r$  is called hyperbolic if its signature is  $(1, r - 1)$ . Even hyperbolic  $p$ -elementary lattices were classified by [RS]. In this section, we classify even hyperbolic 3-elementary lattices admitting a primitive imbedding in an even unimodular lattice of signature  $(3, 19)$ .

We denote by  $U$  the hyperbolic lattice defined by  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  which is an even unimodular lattice of signature  $(1, 1)$ , and by  $A_m$ ,  $D_n$  or  $E_l$  an even negative definite lattice associated with the Dynkin diagram of type  $A_m$ ,  $D_n$  or  $E_l$  ( $m \geq 1$ ,  $n \geq 4$ ,  $l = 6, 7, 8$ ). For a lattice  $L$  and an integer  $m$ ,  $L(m)$  is the lattice whose bilinear form is the one on  $L$  multiplied by  $m$ .

Let  $L$  be a non-degenerate lattice. Then the bilinear form of  $L$  determines a canonical embedding  $L \subset L^* = \text{Hom}(L, \mathbb{Z})$ . We denote by  $A_L$  the factor group  $L^*/L$  which is a finite abelian group.

Let  $p$  be a prime number. A lattice  $L$  is called  $p$ -elementary if  $A_L \simeq (\mathbb{Z}/p\mathbb{Z})^s$ , where  $s$  is the minimal number of generator of  $A_L$ .

For a  $p$ -elementary lattice we always have the inequality  $s \leq n$ , since  $|L^*/L| = p^s$ ,  $|L^*/pL^*| = p^n$  and  $pL^* \subset L \subset L^*$ .

For example  $A_2$ ,  $E_6$ ,  $E_8$ ,  $E_8(3)$ ,  $U$  and  $U(3)$  are all 3-elementary.

Even hyperbolic 3-elementary lattices were classified as follows.

**Theorem 2.1** ([RS]). An even, indefinite,  $p$ -elementary lattice of rank  $n$  for  $p \neq 2$  and  $n \geq 2$  is uniquely determined by its discriminant (i.e., the number  $s$ ).

For  $p \neq 2$  a hyperbolic lattice corresponding to given value of  $s \leq n$  exist if and only if the following conditions are satisfied:  $n \equiv 0 \pmod{2}$  and

$$\begin{cases} \text{for } s \equiv 0 \pmod{2}, & n \equiv 2 \pmod{4} \\ \text{for } s \equiv 1 \pmod{2}, & p \equiv (-1)^{n/2-1} \pmod{4} \end{cases}.$$

And moreover  $n > s > 0$ , if  $n \not\equiv 2 \pmod{8}$ . ■

Let  $L_{K3}$  be an even unimodular lattice of signature  $(3, 19)$ . It is known that  $L_{K3}$  is isometric to  $U^{\oplus 3} \oplus E_8^{\oplus 2}$  by classification of even unimodular indefinite lattices ([Se]).

**Lemma 2.2.** Let  $S$  be an even hyperbolic 3-elementary lattice admitting a primitive imbedding in  $L_{K3}$ . Let  $T$  be the orthogonal complement of  $S$  in  $L_{K3}$ . Then the next tables give all even hyperbolic 3-elementary lattices admitting a primitive imbedding in  $L_{K3}$  (see [Ni1] Sec.1, part 12°).

| rank $S = 2$ |        |                                       |
|--------------|--------|---------------------------------------|
| $s$          | $S$    | $T$                                   |
| 0            | $U$    | $U^{\oplus 2} \oplus E_8^{\oplus 2}$  |
| 2            | $U(3)$ | $U \oplus U(3) \oplus E_8^{\oplus 2}$ |

| rank $S = 4$ |                   |                                       |
|--------------|-------------------|---------------------------------------|
| $s$          | $S$               | $T$                                   |
| 1            | $U \oplus A_2$    | $U^{\oplus 2} \oplus E_6 \oplus E_8$  |
| 3            | $U(3) \oplus A_2$ | $U \oplus U(3) \oplus E_6 \oplus E_8$ |

| rank $S = 6$ |                              |                                       |
|--------------|------------------------------|---------------------------------------|
| $s$          | $S$                          | $T$                                   |
| 2            | $U \oplus A_2^{\oplus 2}$    | $U^{\oplus 2} \oplus E_6^{\oplus 2}$  |
| 4            | $U(3) \oplus A_2^{\oplus 2}$ | $U \oplus U(3) \oplus E_6^{\oplus 2}$ |

| rank $S = 8$ |                              |  |
|--------------|------------------------------|--|
| $s$          | $S$                          | $T$  |
| 1            | $U \oplus E_6$               | $U^{\oplus 2} \oplus E_8 \oplus A_2$       |
| 3            | $U \oplus A_2^{\oplus 3}$    | $U \oplus U(3) \oplus E_8 \oplus A_2$      |
| 5            | $U(3) \oplus A_2^{\oplus 3}$ | $A_2(-1) \oplus E_6 \oplus A_2^{\oplus 3}$ |
| 7            | $U(3) \oplus E_6^*(3)$       | $A_2(-1) \oplus A_2^{\oplus 6}$            |

| rank $S = 10$ |                              |   |
|---------------|------------------------------|---|
| $s$           | $S$                          | $T$                                     |
| 0             | $U \oplus E_8$               | $U^{\oplus 2} \oplus E_8$               |
| 2             | $U \oplus E_6 \oplus A_2$    | $U \oplus U(3) \oplus E_8$              |
| 4             | $U \oplus A_2^{\oplus 4}$    | $U \oplus U(3) \oplus E_6 \oplus A_2$   |
| 6             | $U(3) \oplus A_2^{\oplus 4}$ | $A_2(-1) \oplus A_2^{\oplus 5}$         |
| 8             | $U \oplus E_8(3)$            | $U(3)^{\oplus 2} \oplus A_2^{\oplus 4}$ |
| 10            | $U(3) \oplus E_8(3)$         | $A_2(-1) \oplus A_2 \oplus E_8(3)$      |

| rank $S = 12$ |                                      |   |
|---------------|--------------------------------------|---|
| $s$           | $S$                                  | $T$                                     |
| 1             | $U \oplus E_8 \oplus A_2$            | $A_2(-1) \oplus E_8$                    |
| 3             | $U \oplus E_6 \oplus A_2^{\oplus 2}$ | $A_2(-1) \oplus E_6 \oplus A_2$         |
| 5             | $U \oplus A_2^{\oplus 5}$            | $A_2(-1) \oplus A_2^{\oplus 4}$         |
| 7             | $U(3) \oplus A_2^{\oplus 5}$         | $U(3)^{\oplus 2} \oplus A_2^{\oplus 3}$ |
| 9             | $U \oplus E_8(3) \oplus A_2$         | $A_2(-1) \oplus E_8(3)$                 |

| rank $S = 14$ |                                      |                                 |
|---------------|--------------------------------------|---------------------------------|
| $s$           | $S$                                  | $T$                             |
| 2             | $U \oplus E_8 \oplus A_2^{\oplus 2}$ | $A_2(-1) \oplus E_6$            |
| 4             | $U \oplus E_6 \oplus A_2^{\oplus 3}$ | $A_2(-1) \oplus A_2^{\oplus 3}$ |
| 6             | $U \oplus A_2^{\oplus 6}$            | $U(3)^2 \oplus A_2^{\oplus 2}$  |
| 8             | $U(3) \oplus A_2^{\oplus 6}$         | no existence                    |

| rank $S = 16$ |                                      |                                 |
|---------------|--------------------------------------|---------------------------------|
| $s$           | $S$                                  | $T$                             |
| 1             | $U \oplus E_8 \oplus E_6$            | $U^{\oplus 2} \oplus A_2$       |
| 3             | $U \oplus E_8 \oplus A_2^{\oplus 3}$ | $A_2(-1) \oplus A_2^{\oplus 2}$ |
| 5             | $U \oplus E_6 \oplus A_2^{\oplus 4}$ | $U(3)^2 \oplus A_2$             |

| rank $S = 18$ |                                      |                   |
|---------------|--------------------------------------|-------------------|
| $s$           | $S$                                  | $T$               |
| 0             | $U \oplus E_8^{\oplus 2}$            | $U^{\oplus 2}$    |
| 2             | $U \oplus E_8 \oplus E_6 \oplus A_2$ | $U \oplus U(3)$   |
| 4             | $U \oplus E_8 \oplus A_2^{\oplus 4}$ | $U(3)^{\oplus 2}$ |

| rank $S = 20$ |                                      |           |
|---------------|--------------------------------------|-----------|
| $s$           | $S$                                  | $T$       |
| 1             | $U \oplus E_8^{\oplus 2} \oplus A_2$ | $A_2(-1)$ |

Table 1:

*Proof.* Since  $L_{K3}$  is a unimodular lattice and  $S$  is a primitive sublattice of  $L_{K3}$ ,  $A_S \simeq A_T$  ([BHPV] Lemma 2.5). Hence it is the same that the minimal number of generators of  $A_S$  and  $A_T$ .

By Theorem 2.1, if rank  $S = 2, 6, 10, 14, 18$  then  $s \equiv 0 \pmod{2}$  and if rank  $S = 4, 8, 12, 16, 20$  then  $s \equiv 1 \pmod{2}$ . Moreover if rank  $S = 6, 14$  then  $s > 0$ .

Finally we remark that there is no even unimodular lattice with signature  $(2, 6)$  by classification ([Se]). Hence there are no primitive imbedding of  $U(3) \oplus A_2^{\oplus 6}$  in  $L_{K3}$ .  $\square$

### 3. AUTOMORPHISMS OF ORDER 3 OF LATTICES

Let  $(S, T)$  be as in Lemma 2.2. In this section, we shall show that if  $22 - \rho - 2s < 0$  then there exist no isometries  $f$  of order 3 with the following two properties:

$$(\mathcal{A}) \quad f|_S = \text{id}_S$$

$$(\mathcal{B}) \quad f \text{ has no non-trivial fixed vectors in } T \otimes \mathbb{Q}.$$

(see Theorem 3.3.)

**Lemma 3.1.** Let  $L'$  be an even lattice. And let  $L = L'(3)$ . Then  $L$  has no isometries of order 3 which act trivially on  $A_L$ .

*Proof.* Let  $f : L \rightarrow L$  be an isometry of order 3 which act trivially on  $A_L$ . Since the induced isometry  $A_L \rightarrow A_L$  ( $\bar{x} \mapsto \overline{f^*(x)}$ ) is identity, for all  $x \in L^*$ , there exists an  $l \in L$  such that  $f^*(x) = x + l$ . Since  $f$  is an isometry of order 3, we have  $\langle x, f^*(x) \rangle = \langle f^*(x), f^{*2}(x) \rangle = \langle f^{*2}(x), x \rangle$  and  $f^{*2} + f^* + \text{id}_{L^*} = 0$ . These imply

$$\begin{aligned} 0 &= \langle f^{*2}(x) + f^*(x) + x, x \rangle = \langle f^{*2}(x), x \rangle + \langle f^*(x), x \rangle + \langle x, x \rangle \\ &= 2\langle f^*(x), x \rangle + \langle x, x \rangle \\ &= 2(\langle x, x \rangle + \langle l, x \rangle) + \langle x, x \rangle. \end{aligned}$$

Thus we have

$$(1) \quad 3\langle x, x \rangle = -2\langle l, x \rangle.$$

On the other hand,  $\langle x, x \rangle = \langle f^*(x), f^*(x) \rangle = \langle x + l, x + l \rangle = \langle x, x \rangle + 2\langle x, l \rangle + \langle l, l \rangle$ . Hence

$$(2) \quad -2\langle x, l \rangle = \langle l, l \rangle.$$

It follows from (1) and (2) that  $3\langle x, x \rangle = \langle l, l \rangle$ . Note that  $\langle l, l \rangle \in 6\mathbb{Z}$  by the assumption of  $L$ . Hence for all  $x \in L^*$ ,  $\langle x, x \rangle \in 2\mathbb{Z}$ . This implies that  $L$  is an unimodular lattice. This is a contradiction.  $\square$

**Remark 3.2.**  $U(3) \oplus U$  and  $U \oplus U$  have an isometry of order 3 which acts trivially on the discriminant group.

Actually, let  $e_1, e_2$  (resp.  $e'_1, e'_2$ ) be a basis of  $U(3)$  (resp.  $U$ ) with  $e_1^2 = e_2^2 = 0$ ,  $\langle e_1, e_2 \rangle = 3$  (resp.  $(e'_1)^2 = (e'_2)^2 = 0$ ,  $\langle e'_1, e'_2 \rangle = 1$ ). Let  $\rho_1$  be an isometry of  $U(3) \oplus U$  defined by

$$\begin{aligned} \rho_1(e_1) &= -2e_1 + 3e'_1, & \rho_1(e_2) &= e_2 + 3e'_2 \\ \rho_1(e'_1) &= -e_1 + e'_1, & \rho_1(e'_2) &= -e_2 - 2e'_2. \end{aligned}$$

We can easily see that  $\rho_1$  is an isometry of order 3 which acts trivially on the discriminant group of  $U(3) \oplus U$ .

Similarly, let  $e_1, e_2, e'_1, e'_2$  be a basis of  $U \oplus U$  with  $e_1^2 = e_2^2 = 0$ ,  $\langle e_1, e_2 \rangle = 1$ ,  $(e'_1)^2 = (e'_2)^2 = 0$ ,  $\langle e'_1, e'_2 \rangle = 1$ ,  $\langle e_i, e'_j \rangle = 0$  for all  $i, j = 1, 2$ . Let  $\rho_2$  be an isometry of  $U \oplus U$  defined by

$$\begin{aligned}\rho_2(e_1) &= e_1 + e'_1, & \rho_2(e_2) &= -2e_2 + 3e'_2 \\ \rho_2(e'_1) &= -3e_1 - 2e'_1, & \rho_2(e'_2) &= -e_2 + e'_2.\end{aligned}$$

Note that  $\rho_2$  is an isometry of order 3.

**Theorem 3.3.** Let  $(S, T)$  be as in Lemma 2.2 and let  $\rho = \text{rank } S$ . If  $22 - \rho - 2s < 0$  then  $T$  has no isometries of order 3 with properties  $(\mathcal{A})$ ,  $(\mathcal{B})$ .

*Proof.* We remark that  $U(3)^{\oplus 2}$  and  $E_8(3)$  have no isometries of order 3 which act trivially on the discriminant group by Lemma 3.1. Thus from the tables in Section 2, it is sufficient to prove that  $U(3)^{\oplus 2} \oplus A_2^{\oplus 4}$ ,  $A_2(-1) \oplus A_2 \oplus E_8(3)$ ,  $U(3)^{\oplus 2} \oplus A_2^{\oplus 3}$ ,  $A_2(-1) \oplus E_8(3)$ ,  $U(3)^{\oplus 2} \oplus A_2^{\oplus 2}$  and  $U(3)^{\oplus 2} \oplus A_2$  have no isometries of order 3 with the properties  $(\mathcal{A})$ ,  $(\mathcal{B})$ .

We suppose that  $f$  is an isometry of order 3 with the properties  $(\mathcal{A})$ ,  $(\mathcal{B})$ . By using equation  $f^2 + f + \text{id}_T = 0$ ,

$$\begin{aligned}0 &= \langle f^2(x) + f(x) + x, x \rangle = \langle f^2(x), x \rangle + \langle f(x), x \rangle + \langle x, x \rangle \\ &= 2\langle f(x), x \rangle + \langle x, x \rangle \\ &= 2\langle f(x), x \rangle - 2.\end{aligned}$$

Thus we have  $\langle x, f(x) \rangle = 1$ . Let  $x$  be an element of  $T$  such that  $x^2 = -2$  and let  $M$  be the lattice generated by  $x$  and  $f(x)$ . Obviously  $M \simeq A_2$ .

In the following, for a lattice  $L$ , let  $d(L) = |L^*/L|$ .

(Case 1) Let  $T = U(3)^{\oplus 2} \oplus A_2$  and let  $M'$  be the orthogonal complement of  $M$  in  $T$ . Since  $\text{rank } M' = 4$  and  $d(M') \leq 3^4$ ,  $d(M' \oplus M) \leq 3^5$ . In general if  $N$  is a sublattice of  $N'$  with  $\text{rank } N = \text{rank } N'$ , then  $d(N) = [N' : N]^2 d(N')$ . Therefore it follows that  $d(M' \oplus M) = 3^5$ . Hence  $d(M') = 3^4$ . Thus we have  $M' = U(3)^{\oplus 2}$  by Theorem 2.1. This implies that homomorphism  $f$  induces an isometry of order 3 of  $U(3)^{\oplus 2}$ . This is a contradiction by Lemma 3.1.

Similarly we can see the same assertion for  $A_2(-1) \oplus A_2 \oplus E_8(3)$ ,  $A_2(-1) \oplus E_8(3)$  and  $U(3)^{\oplus 2} \oplus A_2^{\oplus 2}$ .

(Case 2) Next let  $T = U(3)^{\oplus 2} \oplus A_2^{\oplus 3}$ . By the same way as above, we can see that  $d(M' \oplus M) = 3^9$  or  $3^7$ . In case  $d(M' \oplus M) = 3^9$ ,  $M'$  is a lattice with  $\text{rank } M' = 8$ ,  $d(M') = 3^8$  and signature  $(2, 6)$ . But there is

no such lattice by classification of even unimodular lattices [Se]. Thus  $d(M' \oplus M) = 3^7$  and  $M' = U(3)^{\oplus 2} \oplus A_2^{\oplus 2}$ . Then this case reduces to the (Case 1) .

(Case 3) Finally let  $T = U(3)^{\oplus 2} \oplus A_2^{\oplus 4}$ . Then we can see that  $d(M' \oplus M) = 3^{10}$  or  $3^8$ .

If  $d(M' \oplus M) = 3^{10}$  then  $[M'^* \oplus M^* : T^*] = [T : M' \oplus M] = 3$ . For  $(-x - 2f(x))/3 \in M^* = A_2^*$ , there exist  $y \in M'$  such that  $y/3 + (-x - 2f(x))/3 \in T$ . But it follows that  $\langle (y - x - 2f(x))/3, (-x - 2f(x))/3 \rangle = -2/3 \notin \mathbb{Z}$ . Hence  $(-x - 2f(x))/3 \notin T^*$ . Since  $(-x - 2f(x))/3 \in A_2^* \subset T^*$ , this is a contradiction. Thus  $d(M' \oplus M) = 3^8$  and  $M' = U(3)^{\oplus 2} \oplus A_2^{\oplus 3}$ . Then this case reduces to the (Case 2) .  $\square$

#### 4. THE FIXED LOCUS OF NON-SYMPLECTIC AUTOMORPHISMS

In this section, we shall see that the fixed locus  $X^\varphi$  is determined by the invariants of the Néron-Severi lattice  $S_X$ .

**Lemma 4.1.** Let  $X$  be an algebraic  $K3$  surface,  $\varphi$  a non-symplectic automorphism of order 3 on  $X$ . Then we have :

- (1)  $\varphi^* | T_X \otimes \mathbb{C}$  can be diagonalized as:

$$\begin{pmatrix} \zeta I_r & 0 \\ 0 & \bar{\zeta} I_r \end{pmatrix}$$

where  $I_r$  is the identity matrix of size  $r$ ,  $\zeta$  is a primitive third root of unity.

- (2) Let  $P$  be an isolated fixed point of  $\varphi$  on  $X$ . Then  $\varphi^*$  can be written as

$$\begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}$$

under some appropriate local coordinates around  $P$ .

- (3) Let  $C$  be a fixed irreducible curve and  $Q$  a point on  $C$ . Then  $\varphi^*$  can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

under some appropriate local coordinates around  $Q$ . In particular, fixed curves are non-singular.

*Proof.* (1) This follows from [Ni2], Theorem 3.1.

(2), (3) Since  $\varphi^*$  acts on  $H^0(X, \Omega_X^2)$  as a multiplication by  $\zeta$ , it acts on the tangent space of a fixed point as

$$\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}.$$



□

Thus the fixed locus of  $\varphi$  consists of disjoint union of non-singular curves and isolated points. Hence we can express the irreducible decomposition of  $X^\varphi$  as

$$X^\varphi = \{P_1\} \amalg \cdots \amalg \{P_M\} \amalg C_1 \amalg \cdots \amalg C_N,$$

where  $P_j$  is an isolated point and  $C_k$  is a non-singular curve.

**Proposition 4.2.** Let  $\rho$  be a Picard number of  $X$ . Then the number of isolated points  $M$  is  $\rho/2 - 1$ .

*Proof.* First we calculate the holomorphic Lefschetz number  $L(\varphi)$  in two ways as in [AS1], page 542 and [AS2], page 567. That is

$$\begin{aligned} L(\varphi) &= \sum_{i=0}^2 \text{tr}(\varphi^* | H^i(X, \mathcal{O}_X)), \\ L(\varphi) &= \sum_{j=1}^M a(P_j) + \sum_{k=1}^N b(C_k). \end{aligned}$$

Here

$$\begin{aligned} a(P_j) &:= \frac{1}{\det(1 - \varphi^* | T_{P_j})} \\ &= \frac{1}{\det \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix} \right)} \\ &= -\frac{\zeta}{3}, \\ b(C_k) &:= \frac{1 - g(C_k)}{1 - \zeta^{-2}} - \frac{\zeta^{-2} C_k^2}{(1 - \zeta^{-2})^2} \\ &= \frac{\zeta(1 - g(C_k))}{3}, \end{aligned}$$

where  $T_{P_j}$  is the tangent space of  $X$  at  $P_j$ ,  $g(C_k)$  is the genus of  $C_k$  and  $\zeta^2$  is the eigenvalue of the action  $\varphi_*$  on the normal bundle of  $C_k$  (c.f. Lemma 4.1).

Using the Serre duality  $H^2(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X(K_X))^\vee$ , we calculate from the first formula that  $L(\varphi) = 1 + \zeta^2 = -\zeta$ . From the second formula, we obtain

$$L(\varphi) = \frac{-M\zeta}{3} + \sum_{k=1}^N \frac{\zeta(1 - g(C_k))}{3}.$$

Combing two formulae, we have

$$(3) \quad M - \sum_{k=1}^N (1 - g(C_k)) = 3.$$

Next we apply the topological Lefschetz formula:

$$\chi_{\text{top}}(X^\varphi) = \sum_{i=0}^4 (-1)^i \text{tr}(\varphi^* | H^i(X, \mathbb{R})).$$

The left-hand side is

$$(4) \quad \chi_{\text{top}}(X^\varphi) = M + \sum_{k=1}^N (2 - 2g(C_k)).$$

By Lemma 4.1, one has the following diagonalized actions

$$\varphi^* | T_X \otimes \mathbb{C} = \begin{pmatrix} \zeta I_r & 0 \\ 0 & \bar{\zeta} I_r \end{pmatrix}$$

Since  $\varphi^*$  act trivially on  $S_X$ ,  $\text{tr}(\varphi^* | S_X) = \rho$ . Thus we have  $\rho + 2r = 22$ . Hence we can calculate the right-hand side of the Lefschetz formula as follow:

$$\begin{aligned} \sum_{i=0}^4 (-1)^i \text{tr}(\varphi^* | H^i(X, \mathbb{R})) &:= 1 - 0 + \text{tr}(\varphi^* | S_X) + \text{tr}(\varphi^* | T_X) - 0 + 1 \\ &= 2 + \rho - r \\ &= 2 + \rho - \frac{22 - \rho}{2} \\ (5) \quad &= \frac{3\rho - 18}{2}. \end{aligned}$$

By (3), (4) and (5),  $M = \rho/2 - 1$ . □

By the Hodge index theorem, the following three cases are possible:

- (A)  $X^\varphi = \phi$ ;
- (B)  $X^\varphi = \{P_1\} \amalg \cdots \amalg \{P_M\} \amalg C_1^{(1)} \amalg \cdots \amalg C_L^{(1)} \amalg E_1 \amalg \cdots \amalg E_K$ ;
- (C)  $X^\varphi = \{P_1\} \amalg \cdots \amalg \{P_M\} \amalg C^{(g)} \amalg E_1 \amalg \cdots \amalg E_{N-1}$ ,

where we denote by  $P_i$  an isolated point,  $C^{(g)}$  a non-singular curve of genus  $g$  and by  $E_j$  a non-singular rational curve.

The following lemma follows from [PS] §3 Corollary 3 and classification of singular fibers of elliptic fibrations [Kod].

**Lemma 4.3.** Let  $X$  be an algebraic  $K3$  surface. Assume that  $S_X = U(m) \oplus K_1 \oplus \cdots \oplus K_r$ , where  $m = 1$  or  $3$ , and  $K_i$  is a lattice isomorphic to  $A_2$ ,  $E_6$  and  $E_8$ . Then there exist an elliptic fibration  $\pi : X \longrightarrow \mathbb{P}^1$ . Moreover  $\pi$  has a reducible singular fiber whose dual graph is type  $\widetilde{K}_i$ .

**Remark 4.4.** Let  $\{e, f\}$  be a basis of  $U$  (resp.  $U(3)$ ) with  $\langle e, e \rangle = \langle f, f \rangle = 0$  and  $\langle e, f \rangle = 1$  (resp.  $\langle e, f \rangle = 3$ ). If necessary replacing  $e$  by  $\varphi(e)$  is a composition of reflections induces from non-singular rational curves on  $X$ , we may assume that  $e$  is represented by the class of an elliptic curve  $F$  and the linear system  $|F|$  defines an elliptic fibration  $\pi : X \longrightarrow \mathbb{P}^1$ .

We remark that  $X$  has an elliptic fibration from the Table 1. In the following, we consider the elliptic fibration  $\pi : X \longrightarrow \mathbb{P}^1$  by Remark 4.4.

**Lemma 4.5.** In the case (C), if  $\rho < 8$  then the type of singular fiber is only of type II or of type IV.

*Proof.* We obtain  $\chi_{\text{top}}(\sum P_j) = \rho/2 - 1$  and  $\chi_{\text{top}}(X^\varphi) = (3\rho - 18)/2$  by Proposition 4.2 and the topological Lefschetz formula.

From these two equations, we get

$$(6) \quad \chi_{\text{top}} \left( C^{(g)} \amalg \sum_{k=1}^{N-1} E_k \right) = \rho - 8.$$

The left-hand side is calculated as

$$(7) \quad \chi_{\text{top}} \left( C^{(g)} \amalg \sum_{k=1}^{N-1} E_k \right) = (2 - 2g) + 2(N - 1).$$

Therefore if  $\rho < 8$  then there exists a non-singular curve  $C^{(g)}$  with  $g \geq 2$ .

Since  $S_X = U(m) \oplus A_2^l$ , where  $m = 1$  or  $3$ ,  $0 \leq l \leq 3$ , the type of singular fiber of  $\pi$  is of type  $I_1$ , of type II, of type  $I_3$  or of type IV. Let  $F$  be a singular fiber of  $\pi$ . By the Hodge index theorem, the intersection number  $C^{(g)} \cdot F$  is positive. This implies that the automorphism  $\varphi$  acts trivially on the base. Hence a smooth fiber has an automorphism of order 3. Thus a fixed locus of a smooth fiber is exactly three isolated points. Therefore the functional invariant of  $\pi$  is 0. This implies that if  $\rho < 8$  then the type of singular fiber is only of type II and of type IV.  $\square$

First we take up the case (A) and the case (B).

**Lemma 4.6.** The case (A) does not occur, that is,  $X^\varphi \neq \phi$ .

*Proof.* By Proposition 4.2, if  $X^\varphi = \phi$  then  $\rho = 2$ . But if  $\rho = 2$  then  $\chi_{\text{top}}(X^\varphi) = -6$  by the topological Lefschetz formula. This is a contradiction by  $\chi_{\text{top}}(\phi) = 0$ . Hence the case (A) is not realized.  $\square$

**Lemma 4.7.** In the case (B),  $L \leq 1$ .

*Proof.* An elliptic curve  $C_j^{(1)}$  belongs to one elliptic pencil  $|C_j^{(1)}| : X \rightarrow \mathbb{P}^1$ . Assume  $L \geq 2$  and  $K > 0$ . Then  $\varphi$  fixes at least three fibers:  $C_1^{(1)}, \dots, C_L^{(1)}$  and a fiber containing  $E_1$ . Since an automorphism of order 3 on  $\mathbb{P}^1$  has exactly two isolated fixed points,  $\varphi$  is trivial on the base  $\mathbb{P}^1$ . And it is also trivial on a fiber  $C_j^{(1)}$ . Hence  $\varphi$  is a symplectic automorphism. This is a contradiction.

We remark that if  $L = 2$  and  $K = 0$  then  $M > 0$  by Proposition 4.2. Actually if  $\rho = 2$  then  $\chi_{\text{top}}(X^\varphi) = -6$  by the topological Lefschetz formula. This implies that  $X^\varphi$  has a non-singular curve  $C^{(g)}$  with genus  $g \geq 2$ . Hence  $\varphi$  fixes at least three fibers:  $C_1^{(1)}, C_2^{(1)}$  and a fiber containing  $P_1, \dots, P_M$ . This implies that  $\varphi$  is trivial on the base  $\mathbb{P}^1$ . Therefore the case of  $L = 2$  and  $K = 0$  does not occur.  $\square$

The Lemma 4.7 implies that the case (B) is a special case of the case (C). Hence more generally, we take up the case (C). Actually we have the following results.

**Theorem 4.8.** Let  $S_X$  be the Néron-Severi lattice except  $U(3) \oplus E_6^*(3)$ , let  $\rho$  be the Picard number of  $X$  and let  $s$  be the minimal number of generators of  $S_X^*/S_X$ .

- (1) If  $22 - \rho - 2s < 0$ , then  $X$  has no non-symplectic automorphism of order 3 which acts trivially on  $S_X$ .
- (2) If  $22 - \rho - 2s \geq 0$ , then  $X$  has a non-symplectic automorphism  $\varphi$  of order 3 which act trivially on  $S_X$ . Moreover the fixed locus  $X^\varphi$  has the form

$$X^\varphi = \{P_1\} \amalg \dots \amalg \{P_M\} \amalg C^{(g)} \amalg E_1 \amalg \dots \amalg E_{N-1}$$

where  $M = \rho/2 - 1$ ,  $g = (22 - \rho - 2s)/4$ ,  $N = (6 + \rho - 2s)/4$ .

*Proof.* (1) This follows from Theorem 3.3.

(2)  $M = \rho/2 - 1$  follows from Proposition 4.2. We prove  $g = (22 - \rho - 2s)/4$  and  $N = (6 + \rho - 2s)/4$ .

(Case 1)  $\rho < 8$

Let  $F$  be a fiber of  $\pi$ . From the proof of Lemma 4.5,  $\varphi$  acts trivially on the base of  $\pi$ . Thus if  $\pi$  has a section (resp. no section) then  $C^{(g)}.F = 2$  (resp.  $C^{(g)}.F = 3$ ) in this case.

It is known that

$$(8) \quad \sum_{F: \text{ singular fiber}} \chi_{\text{top}}(F) = \chi_{\text{top}}(X) = 24.$$

Since the Euler number of a singular fiber of type II or IV is 2 or 4, respectively, the number of singular fiber of type IV is  $[24 - 4\{(\rho - 2)/2\}]/2 = 14 - \rho$  by equation (8). If  $\pi$  has a section (resp. no section) then the number of singular fiber of type IV =  $s$  (resp.  $s - 2$ ). Thus  $(\rho - 2)/2 = s$  (resp.  $(\rho - 2)/2 = s - 2$ ).

We remark that  $C^{(g)}$  meets a singular fiber of type II at cusp. Since if  $\pi$  has a section then  $C^{(g)}.F = 2$ , by the Hurwitz formula, we have

$$\begin{aligned} 2g - 2 &= 2(2g(\mathbb{P}^1) - 2) + (14 - \rho)(2 - 1) \\ 2g &= \frac{24 - 2\rho}{2} \\ &= \frac{24 - \rho - (2s - 2)}{2} \\ g &= \frac{22 - \rho - 2s}{4}. \end{aligned}$$

Next we assume  $\pi$  has no sections.

By Lemma 4.5, if  $X^\varphi$  contains some non-singular rational curves  $E_j$  then  $E_j$  are components of singular fiber of type IV. But this is a contradiction by  $C^{(g)}.F = 3$ . Hence  $X^\varphi$  contains no non-singular rational curve. By the Lefschetz formula,

$$\begin{aligned} (2 - 2g) + \frac{\rho - 2}{2} &= \frac{3\rho - 18}{2} \\ 2g &= \frac{20 - 2\rho}{2} \\ &= \frac{20 - \rho - (2s - 2)}{2} \\ g &= \frac{22 - \rho - 2s}{4}. \end{aligned}$$

Moreover by this equation, (6) and (7), we have

$$N = \frac{6 + \rho - 2s}{4}.$$

(Case 2)  $\rho \geq 8$  and  $\pi$  has a section.

In this case, we argue the assertion each  $S_X$ . The following are how to calculate the genus and the number of fixed curve.

Assume  $S_X = U \oplus E_8 \oplus E_6$ . By (6) and (7),  $g = N - 4$ . Since  $\text{rank } S_X = 16$ , we have  $\chi_{\text{top}} \left( C^{(g)} \amalg \sum_{k=1}^{N-1} E_k \right) = 8$ . Thus the number of non-singular rational curves which  $X^\varphi$  contains is 4 or more.

Now  $\pi$  has a singular fiber of type  $\text{II}^*$  and  $\text{IV}^*$ . Since the component with multiplicity 6 of singular fiber of type  $\text{II}^*$  and the component with multiplicity 3 of singular fiber of type  $\text{IV}^*$  are pointwisely fixed by  $\varphi$ , if  $g = 0$  (i.e.  $N = 4$ ) then the number of isolated fixed points of  $\varphi$  is exactly eight. This is a contradiction by Proposition 4.2.

If  $g \geq 2$  then the automorphism  $\varphi$  acts trivially on the base of an elliptic fibration. Since the Euler number of a singular fiber of type  $\text{II}^*$  and  $\text{IV}^*$  is 10 or 8, respectively,  $\pi$  has three singular fibers of type  $\text{II}$ . By the Hurwitz formula, we have

$$2g - 2 = 2(2g(\mathbb{P}^1) - 2) + 4(2 - 1).$$

But this is a contradiction by  $g \geq 2$ . Hence  $g = 1$  and  $N = 5$ . Therefore  $X^\varphi := C^{(1)} \amalg \coprod_{i=1}^4 \mathbb{P}_i^1 \amalg \coprod_{j=1}^7 \{P_j\}$ .

Similarly in other cases we can calculate the genus and the numbers of fixed curve by the same argument of these examples. And these results satisfies the assertion.

(Case 3)  $\rho \geq 8$  and  $\pi$  has no section.

Assume  $S_X = U(3) \oplus A_2^4$ . By (6) and (7),  $g = N - 1$ . Since  $\text{rank } S_X = 10$ ,  $\chi_{\text{top}} \left( C^{(g)} \amalg \sum_{k=1}^{N-1} E_k \right) = 2$ . Thus the number of non-singular rational curves which  $X^\varphi$  contains is one or more.

Now  $\pi$  has four singular fibers  $F_i$  of type  $\text{IV}$  or  $\text{I}_3$ . Thus automorphism  $\varphi$  acts trivially on the base of an elliptic fibration. Since  $\pi$  has no section,  $C^{(g)}.F_i = 3$ . Moreover we can see that all  $F_i$  are singular fiber of type  $\text{IV}$  by Lemma 4.3. And a singular fiber of type  $\text{IV}$  has exactly one isolated fixed point at center. Therefore  $\varphi$  has exactly one fixed curve. Hence  $X^\varphi := C^{(0)} \amalg \{P_1\} \amalg \{P_2\} \amalg \{P_3\} \amalg \{P_4\}$ .

Similarly we can see that same assertion for other cases.  $\square$

Finally we consider the case of  $S_X = U(3) \oplus E_6^*(3)$ .

**Proposition 4.9.** If  $S_X = U(3) \oplus E_6^*(3)$  then  $X^\varphi = \{P_1\} \amalg \{P_2\} \amalg \{P_3\}$ .

*Proof.* First we show that  $S_X$  has no  $(-2)$  vectors. Let  $q_{E_6}$  be a discriminant form of  $E_6$  and let  $\bar{a}$  be a generator of  $E_6^*/E_6$ . Since  $U \oplus E_6^*/U \oplus E_6 \simeq E_6^*/E_6$  and  $q_{E_6}(\bar{a}) = -4/3$ ,  $U \oplus E_6^*$  has no  $(-2/3)$  vectors. This implies that  $S_X$  has no  $(-2)$  vectors. Hence  $X$  has no non-singular rational curve, i.e.  $K = 0$ .

Since  $\rho = 8$ , we have  $M = 3$  by Proposition 4.2. From (6), (7) and Lemma 4.7, if there exists a fixed curve  $C^{(g)}$  then  $g = 1$ .

Now we assume that  $X^\varphi$  contains fixed curve, i.e.  $X^\varphi = \{P_1\} \amalg \{P_2\} \amalg \{P_3\} \amalg C^{(1)}$ . Let  $\{e, f\}$  be a basis of  $U(3)$ . By Remark 4.4,  $|e|$  define an elliptic fibration  $\pi : X \longrightarrow \mathbb{P}^1$ . We assume that  $\varphi$  acts trivially on the base of  $\varphi$ . Then every smooth fiber of  $\varphi$  has an automorphisms of order 3, and hence the functional invariant is 0. Moreover  $C^{(1)}$  meets a smooth fiber at three points. If  $\varphi$  has a reducible fiber then  $X$  has a non-singular rational curve as a component of the reducible fiber. This is a contradiction. Hence  $\pi$  has 12 singular fibers of type II by (8). Obviously 12 cusps of 12 singular fibers are contained in  $X^\varphi$ . Hence at least 9 cusps lie on  $C^{(1)}$ . Then  $\pi|_{C^{(1)}} : C^{(1)} \longrightarrow \mathbb{P}^1$  is a covering of degree 3 ramified at these 9 cusps. On the other hand, the Hurwitz formula implies that

$$0 = 2g(C^{(1)}) - 2 \geq 3(2g(\mathbb{P}^1) - 2) + 9(2 - 1) = 3.$$

This is a contradiction.

Next we assume that  $\varphi$  acts on the base of  $\varphi$  as an automorphism of order 3. Then  $\varphi$  has exactly two isolated fixed points  $Q_1$  and  $Q_2$  on the base of  $\varphi$ . The  $C^{(1)}$  is equal to  $\varphi^{-1}(Q_1)$  or  $\varphi^{-1}(Q_2)$ . Since  $C^{(1)}.F = e.f = 3$  and  $C^{(1)}$  is fixed by  $\varphi$ ,  $\varphi$  acts trivially on the base of  $|f|$  where  $F$  is a smooth fiber of  $|f|$ . Hence this case reduces to the argument as above.

Thus  $X^\varphi$  has no fixed curve. Hence  $X^\varphi = \{P_1\} \amalg \{P_2\} \amalg \{P_3\}$ .  $\square$

Theorem 1.2 follows from Theorem 4.8 and Proposition 4.9.

**Remark 4.10.** The case of  $S_X = U(3) \oplus E_6^*(3)$  satisfy equation about the number of fixed curves : $N$  in Theorem 4.8. Indeed since  $\rho = 8$  and  $s = 7$ ,  $N = 0$ .

The following is a correspondence of  $S_X$  and  $X^\varphi$ .

| $S_X$                        | $X^\varphi$   |
|------------------------------|---|
| $U$                          | $C^{(5)} \amalg \mathbb{P}^1$                                 |
| $U(3)$                       | $C^{(4)}$   |
| $U \oplus A_2$               | $C^{(4)} \amalg \mathbb{P}^1 \amalg \{pt\}$                   |
| $U(3) \oplus A_2$            | $C^{(3)} \amalg \{pt\}$                                       |
| $U \oplus A_2^{\oplus 2}$    | $C^{(3)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 2$          |
| $U(3) \oplus A_2^{\oplus 2}$ | $C^{(2)} \amalg \{pt\} \times 2$                              |
| $U \oplus E_6$               | $C^{(3)} \amalg \mathbb{P}^1 \times 2 \amalg \{pt\} \times 3$ |
| $U \oplus A_2^{\oplus 3}$    | $C^{(2)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 3$          |
| $U(3) \oplus A_2^{\oplus 3}$ | $C^{(1)} \amalg \{pt\} \times 3$                              |
| $U(3) \oplus E_6^*(3)$       | $\{pt\} \times 3$   |
| $U \oplus E_8$               | $C^{(3)} \amalg \mathbb{P}^1 \times 3 \amalg \{pt\} \times 4$ |

|                                      |   |
|--------------------------------------|---|
| $U \oplus E_6 \oplus A_2$            | $C^{(2)} \amalg \mathbb{P}^1 \times 2 \amalg \{pt\} \times 4$ |
| $U \oplus A_2^{\oplus 4}$            | $C^{(1)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 4$          |
| $U(3) \oplus A_2^{\oplus 4}$         | $C^{(0)} \amalg \{pt\} \times 4$                              |
| $U \oplus E_8 \oplus A_2$            | $C^{(2)} \amalg \mathbb{P}^1 \times 3 \amalg \{pt\} \times 5$ |
| $U \oplus E_6 \oplus A_2^{\oplus 2}$ | $C^{(1)} \amalg \mathbb{P}^1 \times 2 \amalg \{pt\} \times 5$ |
| $U \oplus A_2^{\oplus 5}$            | $C^{(0)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 5$          |
| $U \oplus E_8 \oplus A_2^{\oplus 2}$ | $C^{(1)} \amalg \mathbb{P}^1 \times 3 \amalg \{pt\} \times 6$ |
| $U \oplus E_6 \oplus A_2^{\oplus 3}$ | $C^{(0)} \amalg \mathbb{P}^1 \times 2 \amalg \{pt\} \times 6$ |
| $U \oplus E_8 \oplus E_6$            | $C^{(1)} \amalg \mathbb{P}^1 \times 4 \amalg \{pt\} \times 7$ |
| $U \oplus E_8 \oplus A_2^{\oplus 3}$ | $C^{(0)} \amalg \mathbb{P}^1 \times 3 \amalg \{pt\} \times 7$ |
| $U \oplus E_8^{\oplus 2}$            | $C^{(1)} \amalg \mathbb{P}^1 \times 5 \amalg \{pt\} \times 8$ |
| $U \oplus E_8 \oplus E_6 \oplus A_2$ | $C^{(0)} \amalg \mathbb{P}^1 \times 4 \amalg \{pt\} \times 8$ |
| $U \oplus E_8^{\oplus 2} \oplus A_2$ | $C^{(0)} \amalg \mathbb{P}^1 \times 5 \amalg \{pt\} \times 9$ |

Table 2: Néron-Severi lattices and fixed locus

## 5. EXAMPLES

In this section, we give examples of algebraic  $K3$  surfaces with non-symplectic automorphism of order 3. Moreover we examine fixed locus of the automorphism.

**Example 5.1.** In the following we give affine equations of elliptic  $K3$  surfaces. We define an automorphism  $\varphi$  of  $X$  as follows:  $\varphi(x, y, z, u) = (x, y, \zeta z, u)$  where  $\zeta$  is a primitive third root of unity.

| $S_X$                     | definition equation  |
|---------------------------|--|
| $U$                       | $z^3 = y \left( y^2 \prod_{i=1}^{12} (u - a_i) - x^2 \right)$                        |
| $U \oplus A_2$            | $z^3 = y \left( y^2 (u - a_0)^2 \prod_{i=1}^{10} (u - a_i) - x^2 \right)$            |
| $U \oplus A_2^{\oplus 2}$ | $z^3 = y \left( y^2 (u - a_0)^2 \prod_{i=1}^8 (u - a_i) (u - a_9)^2 - x^2 \right)$   |
| $U \oplus E_6$            | $z^3 = y \left( y^2 (u - a_0)^4 \prod_{i=1}^8 (u - a_i) - x^2 \right)$               |
| $U \oplus A_2^{\oplus 3}$ | $z^3 = y \left( y^2 \prod_{i=1}^6 (u - a_i) \prod_{j=7}^9 (u - a_j)^2 - x^2 \right)$ |



|                                      |   |
|--------------------------------------|---|
| $U \oplus E_8$                       | $z^3 = y \left( y^2(u - a_0)^5 \prod_{i=1}^7 (u - a_i) - x^2 \right)$                           |
| $U \oplus E_6 \oplus A_2$            | $z^3 = y \left( y^2(u - a_0)^4 \prod_{i=1}^6 (u - a_i)(u - a_7)^2 - x^2 \right)$                |
| $U \oplus A_2^{\oplus 4}$            | $z^3 = y \left( y^2 \prod_{i=1}^4 (u - a_i) \prod_{j=5}^8 (u - a_j)^2 - x^2 \right)$            |
| $U \oplus E_8 \oplus A_2$            | $z^3 = y \left( y^2(u - a_0)^5 \prod_{i=1}^5 (u - a_i)(u - a_6)^2 - x^2 \right)$                |
| $U \oplus E_6 \oplus A_2^{\oplus 2}$ | $z^3 = y \left( y^2(u - a_0)^4 \prod_{i=1}^4 (u - a_i) \prod_{i=5}^6 (u - a_i)^2 - x^2 \right)$ |
| $U \oplus A_2^{\oplus 5}$            | $z^3 = y \left( y^2 \prod_{i=1}^2 (u - a_i) \prod_{i=3}^7 (u - a_i)^2 - x^2 \right)$            |
| $U \oplus E_8 \oplus A_2^{\oplus 2}$ | $z^3 = y \left( y^2(u - a_0)^5 \prod_{i=1}^3 (u - a_i) \prod_{i=4}^5 (u - a_i)^2 - x^2 \right)$ |
| $U \oplus E_6 \oplus A_2^{\oplus 3}$ | $z^3 = y \left( y^2(u - a_0)^4 \prod_{i=1}^3 (u - a_i)^2 \prod_{j=4}^5 (u - a_j) - x^2 \right)$ |
| $U \oplus E_8 \oplus E_6$            | $z^3 = y \left( y^2(u - a_0)^5 \prod_{i=1}^3 (u - a_i)(u - a_4)^4 - x^2 \right)$                |
| $U \oplus E_8 \oplus A_2^{\oplus 3}$ | $z^3 = y \left( y^2(u - a_0)^5 \prod_{i=1}^3 (u - a_i)^2 (u - a_4) - x^2 \right)$               |
| $U \oplus E_8^{\oplus 2}$            | $z^3 = y \left( y^2 \prod_{i=1}^2 (u - a_i)^5 \prod_{j=3}^4 (u - a_j) - x^2 \right)$            |
| $U \oplus E_8 \oplus E_6 \oplus A_2$ | $z^3 = y (y^2(u - a_0)^5 (u - a_1)^4 (u - a_2)^2 (u - a_3) - x^2)$                              |
| $U \oplus E_8^{\oplus 2} \oplus A_2$ | $z^3 = y \left( y^2(u - a_0)^2 \prod_{i=1}^2 (u - a_i)^5 - x^2 \right)$                         |

Table 3: Néron-Severi lattices and definition equations

Next we study a example in Example 5.1 in detail.

**Case:**  $S_X = U$  ([Kon1]) Let  $[x : y : z]$  be a system of a homogeneous coordinate of  $\mathbb{P}^2$ . We take two copies  $W_0 := \mathbb{P}^2 \times \mathbb{C}_1$  and  $W_1 := \mathbb{P}^2 \times \mathbb{C}_2$  of the cartesian product  $\mathbb{P}^2 \times \mathbb{C}$  and form their union  $W = W_0 \cup W_1$  by identifying  $([x : y : z], u) \in W_0$  with  $([x_1 : y_1 : z_1], u_1) \in W_1$  if and only

if  $u = 1/u_1$ ,  $x = x_1$ ,  $u = u_1^6 y_1$  and  $z = u_1^2 z_1$ . we define a subvariety  $X$  of  $W$  by the following equations:

$$(*) \quad \begin{cases} z_1^3 - y_1 (y_1^2 \prod_{i=1}^{12} (u_1 - a_i) - x_1^2) = 0, \\ z_2^3 - y_2 (y_2^2 \prod_{i=1}^{12} (1 - u_2 a_i) - x_2^2) = 0 \end{cases}$$

where  $a_i$  ( $i = 1, 2, \dots, 12$ ) are distinct complex numbers.

Let  $\pi$  be a projection from  $X$  to the  $u$ -sphere  $\mathbb{P}^1$ . It is easy to see that  $X$  is  $K3$  surface and  $\pi^{-1}(u)$  is a non-singular elliptic curve with the functional invariant 0 for ever  $u$  except  $a_i$  ( $i = 1, \dots, 12$ ). Moreover we can see that  $\pi^{-1}(a_i)$  is singular fiber of type II. We define an automorphism  $\varphi$  of  $X$  as follows:  $\varphi([x : y : z], u) = ([x : y : \zeta z], u)$  where  $\zeta$  is a primitive 3-th root of unity. Obviously  $\varphi$  is of order 3. Now we remark that  $X$  has a section  $E$  defined by  $y = 0$ . Let  $F$  be an class of general elliptic curves. Then  $E, F$  generate the Néron-Severi lattice of  $X$  isometric to  $U$ .

By the way, the fixed locus of  $\varphi$  is the set of  $\{z_1 = 0\}$ . That is  $\{y_1 = 0\} \amalg \{y_1^2 \prod_{i=1}^{12} (u_1 - a_i) - x_1^2 = 0\}$ . Clearly, the genus of the curve defined by  $\{y_1 = 0\}$  is 0. Hence this curve is a section of the elliptic fibration.

Let  $C$  be the curve defined by  $y_1^2 \prod_{i=1}^{12} (u_1 - a_i) - x_1^2 = 0$ . The automorphism  $\varphi$  induces an automorphism of order 3 on  $E$  and  $\pi^{-1}(a_i)$  ( $i = 1, \dots, 12$ ). Since  $\varphi$  preserves a cusp of  $\pi^{-1}(a_i)$ , the curve  $C$  and  $\pi^{-1}(a_i)$  intersect at the cusp of  $\pi^{-1}(a_i)$ . Thus we calculate the genus of  $C$  by the Hurwitz formula:

$$2g(C) - 2 = 2(2g(\mathbb{P}^1) - 2) + 12(2 - 1).$$

Therefore we can express the fixed locus  $X^\varphi$  as  $C^{(5)} \amalg \mathbb{P}^1$ .

The next two examples are explained in detail by [Kon2].

**Example 5.2.** Let  $C$  be a smooth non-hyperelliptic curve of genus 4. Then its canonical model is the complete intersection of an irreducible quadric surface  $Q$  and an irreducible cubic surface  $S$  in  $\mathbb{P}^3$ . Let  $X$  be the triple cover of  $Q$  branched along  $C$ . Then  $X$  is a  $K3$  surface with an automorphism  $\varphi$  of order 3 and  $C$  is a fix curve of  $\varphi$ . Since  $\varphi$  has a fixed curve  $C$ ,  $\varphi$  acts on  $H^0(X, \Omega_X^2)$  as a multiplication by third root of unity ([Ni2], §5). Hence  $\varphi$  is a non-symplectic automorphism.

Let  $E$  (resp.  $F$ ) be the inverse image of a smooth fiber of one of the rulings of  $Q$  (resp. another ruling of  $Q$ ). Then  $E, F$  are elliptic curve with  $\langle E, F \rangle = 3$  and  $E, F$  generate the Néron-Severi lattice of  $X$  isometric to  $U(3)$ .

Since  $S_X = U(3)$  contains no  $(-2)$  vectors,  $X$  has no non-singular rational curves. And by the topological Lefschetz formula,  $\chi_{\text{top}}(X^\varphi) = -6 = \chi_{\text{top}}(C)$ . Hence  $X^\varphi$  contains no isolated points. Thus the fixed locus of  $\varphi$  is only  $C$ . Hence  $X^\varphi = C^{(4)}$ .

**Example 5.3.** Let  $C_1$  be a curve in a smooth quadric  $Q$  of bidegree  $(3, 3)$  with one node  $p$ . Let  $L_1, L_2$  be the two lines through  $p$ . First blow up at  $p$  and denote by  $E$  the exceptional curve. Next blow up the two points in which  $E$  and the proper transform of  $C_1$  meet. Then take the triple cover  $X'$  branched along the proper transform of  $C_1$  and  $E$ . Then  $X'$  contains an exceptional curve of the first kind which is the pullback of the proper transform of  $E$ . By contracting this exceptional curve to a smooth point  $q$ , we have  $K3$  surface  $X_1$ . Now for the generic  $K3$  surface  $X_1$ , the Néron-Severi lattice  $S_{X_1}$  is isometric to  $U(3) \oplus A_2$ .

It is easy to see the fixed locus of  $\varphi$  is the smooth point  $q$  and  $\widetilde{C}_1$  where  $\widetilde{C}_1$  is a smooth curve given by to normalize  $C_1$  at  $p$ . Since the genus of  $\widetilde{C}_1 = 3$ , we have  $X_1^\varphi = \{q\} \amalg C^{(3)}$ .

**Example 5.4.** Let  $C_2$  be a curve in a smooth quadric  $Q$  of bidegree  $(3, 3)$  with two nodes. By the same construction as Example 5.3, we have  $K3$  surface  $X_2$  with  $S_{X_2} \simeq U(3) \oplus A_2^{\oplus 2}$  and non-symplectic automorphism  $\varphi$  of order 3. And it is easy to see  $X_2^\varphi = \{P_1\} \amalg \{P_2\} \amalg C^{(2)}$ .

Similarly let  $C_3$  be a curve in a smooth quadric  $Q$  of bidegree  $(3, 3)$  with three nodes. Then we have  $K3$  surface  $X_3$  with  $S_{X_3} \simeq U(3) \oplus A_2^{\oplus 3}$  and non-symplectic automorphism  $\varphi$  of order 3. And it is easy to see  $X_3^\varphi = \{P_1\} \amalg \{P_2\} \amalg \{P_3\} \amalg C^{(1)}$ .

Finally let  $C_4$  be a curve in a smooth quadric  $Q$  of bidegree  $(3, 3)$  with four nodes. Similarly we have  $K3$  surface  $X_4$  with  $S_{X_4} \simeq U(3) \oplus A_2^{\oplus 4}$  and non-symplectic automorphism  $\varphi$  of order 3. Then we see  $X_4^\varphi = \{P_1\} \amalg \{P_2\} \amalg \{P_3\} \amalg \{P_4\} \amalg C^{(0)}$ .

**Example 5.5** ([Zh]). In this example, we construct a  $K3$  surface  $X$  with  $S_X = U(3) \oplus E_6^*(3)$ .

Denote by  $[x : y : z]$  the homogeneous coordinates of  $\mathbb{P}^2$ . Consider three cubic curves with a cusp of  $\mathbb{P}^2$ :

$$C_1 : x^3 = y^2 z, \quad C_2 : y^3 = z^2 x, \quad C_3 : z^3 = x^2 y.$$

Let  $\eta$  be a primitive 7-th root of unity. Then  $C_1 \cap C_2 \cap C_3 = \{[\eta^{3j} : \eta^j : 1] \mid 0 \leq j \leq 6\}$ . Let  $\pi : Y \longrightarrow \mathbb{P}^2$  be the blow-up at  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$ ,  $[0 : 0 : 1]$ , and seven points of  $C_1 \cap C_2 \cap C_3$ , let  $D_i = \pi^* C_i$ . It is easy to see  $D_i^2 = -3$ ,  $\chi_{\text{top}}(Y) = 13$ , and  $0 = \pi^* \sum_{i=1}^3 (C_i + 3K_{\mathbb{P}^2}) = \sum_{i=1}^3 (D_i + 3K_Y)$ .

Now let  $X$  be the triple cover of  $Y$  branched along the  $D_1$ ,  $D_2$ , and  $D_3$ . Hence  $X$  is a  $K3$  surface with non-symplectic automorphism order 3. And the fixed locus is only three isolated points.

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